

# LOW REGULARITY WELL-POSEDNESS FOR THE 3D KLEIN - GORDON - SCHRÖDINGER SYSTEM

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**ABSTRACT.** The Klein-Gordon-Schrödinger system in 3D is shown to be locally well-posed for Schrödinger data in  $H^s$  and wave data in  $H^\sigma \times H^{\sigma-1}$ , if  $s > -\frac{1}{4}$ ,  $\sigma > -\frac{1}{2}$ ,  $\sigma - 2s > \frac{3}{2}$  and  $\sigma - 2 < s < \sigma + 1$ . This result is optimal up to the endpoints in the sense that the local flow map is not  $C^2$  otherwise. It is also shown that (unconditional) uniqueness holds for  $s = \sigma = 0$  in the natural solution space  $C^0([0, T], L^2) \times C^0([0, T], L^2) \times C^0([0, T], H^{-\frac{1}{2}})$ . This solution exists even globally by Colliander, Holmer and Tzirakis [6]. The proofs are based on new well-posedness results for the Zakharov system by Bejenaru, Herr, Holmer and Tataru [3], and Bejenaru and Herr [4].

## 1. INTRODUCTION AND MAIN RESULTS

We consider the (3+1)-dimensional Cauchy problem for the Klein - Gordon - Schrödinger system with Yukawa coupling

$$i\partial_t u + \Delta u = nu \quad (1)$$

$$\partial_t^2 n + (1 - \Delta)n = |u|^2 \quad (2)$$

with initial data

$$u(0) = u_0, \quad n(0) = n_0, \quad \partial_t n(0) = n_1, \quad (3)$$

where  $u$  is a complex-valued and  $n$  a real-valued function defined for  $(x, t) \in \mathbb{R}^3 \times [0, T]$ . This is a classical model which describes a system of scalar nucleons interacting with neutral scalar mesons. The nucleons are described by the complex scalar field  $u$  and the mesons by the real scalar field  $n$ . The mass of the meson is normalized to be 1.

Our results do not use the energy conservation law but only charge conservation  $\|u(t)\|_{L^2(\mathbb{R}^3)} \equiv \text{const}$  (for the global existence result), so they are equally true if one replaces  $nu$  and  $|u|^2$  by  $-nu$  and/or  $-|u|^2$ , respectively.

We are interested in local and global solutions for data

$$u_0 \in H^s(\mathbb{R}^3), \quad n_0 \in H^\sigma(\mathbb{R}^3), \quad n_1 \in H^{\sigma-1}(\mathbb{R}^3)$$

with minimal  $s$  and  $\sigma$ .

Local well-posedness for data  $u_0 \in L^2(\mathbb{R}^3)$ ,  $n_0 \in L^2(\mathbb{R}^3)$ ,  $n_1 \in H^{-1}(\mathbb{R}^3)$  was shown by the author [13] based on estimates given by Ginibre, Tsutsumi and Velo [7] for the Zakharov system, more precisely these solutions exist uniquely in Bourgain type spaces which are subsets of the natural solution spaces

$$(u, n, \partial_t n) \in C^0([0, T], L^2(\mathbb{R}^3)) \times C^0([0, T], L^2(\mathbb{R}^3)) \times C^0([0, T], H^{-1}(\mathbb{R}^3)). \quad (4)$$

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Colliander, Holmer and Tzirakis [6] proved that under the same assumptions on the data global well-posedness holds in Strichartz-type spaces, especially (4) holds for any  $T > 0$ .

Concerning the closely related wave Schrödinger system local well-posedness was shown for  $s > -\frac{1}{4}$  and  $\sigma > -\frac{1}{38}$  and also global well-posedness for certain  $s, \sigma < 0$  by T. Akahori [1],[2].

Thus three questions arise:

- (a) Can we show local well-posedness for even rougher data ?
- (b) Is it possible to show the sharpness of this local well-posedness result ?
- (c) Under which assumptions on the data can we show unconditional uniqueness, i.e. uniqueness in the natural solution space ?

Concerning (a) we prove that local well-posedness holds in Bourgain type spaces which are subsets of the natural spaces, provided the data fulfill

$$s > -\frac{1}{4}, \sigma > -\frac{1}{2}, \sigma - 2s < \frac{3}{2}, \sigma - 2 < s < \sigma + 1.$$

Especially the choice  $s = -\frac{1}{4} +$  and  $\sigma = -\frac{1}{2} +$  is possible, thus we relax the regularity assumptions for the Schrödinger and wave parameters by almost  $\frac{1}{4}$  and  $\frac{1}{2}$  order of derivatives, respectively.

Concerning (b) the estimates for the nonlinearities which lead to the local well-posedness result fail if  $s < -\frac{1}{4}$  or  $\sigma < -\frac{1}{2}$  or  $\sigma - 2s > \frac{3}{2}$ . Using ideas of Holmer [8] and Bejenaru, Herr, Holmer and Tataru [3] we can even show that the solution map  $(u_0, n_0, n_1) \mapsto (u(t), n(t), \partial_t n(t))$  is not  $C^2$  in these cases, i.e. some type of ill-posedness holds.

Concerning (c) we show that for data  $u_0 \in L^2(\mathbb{R}^3)$ ,  $n_0 \in L^2(\mathbb{R}^3)$ ,  $n_1 \in H^{-1}(\mathbb{R}^3)$  unconditional uniqueness holds in the space (4). Using the global existence result of [6] we get unconditional global well-posedness in this case.

The question of unconditional uniqueness was considered among others by Yi Zhou for the KdV equation [16] and nonlinear wave equations [17], by N. Masmoudi and K. Nakanishi for the Maxwell-Dirac, the Maxwell-Klein-Gordon equations [9], the Klein-Gordon-Zakharov system and the Zakharov system [10], and by F. Planchon [14] for semilinear wave equations.

The results in this paper are based on the (3+1)-dimensional estimates by Bejenaru and Herr [4] which they recently used to show a sharp well-posedness result for the Zakharov system. We also use the corresponding sharp (2+1)-dimensional local well-posedness results for the Zakharov system by Bejenaru, Herr, Holmer and Tataru [3], especially their counterexamples.

We use the standard Bourgain spaces  $X^{m,b}$  for the Schrödinger equation, which are defined as the completion of  $\mathcal{S}(\mathbb{R}^3 \times \mathbb{R})$  with respect to

$$\|f\|_{X^{m,b}} := \|\langle \xi \rangle^m \langle \tau + |\xi|^2 \rangle^b \widehat{f}(\xi, \tau)\|_{L_{\xi\tau}^2}.$$

Similarly  $X_{\pm}^{m,b}$  for the equation  $i\partial_t n_{\pm} \mp A^{1/2} n_{\pm} = 0$  is the completion of  $\mathcal{S}(\mathbb{R}^3 \times \mathbb{R})$  with respect to

$$\|f\|_{X_{\pm}^{m,b}} := \|\langle \xi \rangle^m \langle \tau \pm |\xi| \rangle^b \widehat{f}(\xi, \tau)\|_{L_{\xi\tau}^2}.$$

For a given time interval  $I$  we define  $\|f\|_{X^{m,b}(I)} := \inf_{\tilde{f}|_I = f} \|\tilde{f}\|_{X^{m,b}}$  and similarly  $\|f\|_{X_{\pm}^{m,b}(I)}$ . We often skip  $I$  from the notation.

In the following we mean by a solution of a system of differential equation always a solution of the corresponding system of integral equations.

Before formulating the main results of our paper we recall that the KGS system can be transformed into a first order (in  $t$ ) system as follows: if

$$(u, n, \partial_t n) \in C^0([0, T], H^s) \times C^0([0, T], H^{\sigma}) \times C^0([0, T], H^{\sigma-1})$$

is a solution of (1),(2),(3) with data  $(u_0, n_0, n_1) \in H^s \times H^\sigma \times H^{\sigma-1}$ , then defining  $A := -\Delta + 1$  and

$$n_\pm := n \pm iA^{-\frac{1}{2}}\partial_t n$$

and

$$n_{\pm 0} := n_0 \pm iA^{-\frac{1}{2}}n_1 \in H^\sigma,$$

we get that

$$(u, n_+, n_-) \in C^0([0, T], H^s) \times C^0([0, T], H^\sigma) \times C^0([0, T], H^\sigma)$$

is a solution of the following problem:

$$i\partial_t u + \Delta u = \frac{1}{2}(n_+ - n_-)u \quad (5)$$

$$i\partial_t n_\pm \mp A^{1/2}n_\pm = \pm A^{-1/2}(|u|^2) \quad (6)$$

$$u(0) = u_0, \quad n_\pm(0) = n_{\pm 0} := n_0 \pm iA^{-1/2}n_1. \quad (7)$$

The corresponding system of integral equations reads as follows:

$$u(t) = e^{it\Delta}u_0 + \frac{1}{2} \int_0^t e^{i(t-\tau)\Delta}(n_+(\tau) + n_-(\tau))u(\tau)d\tau \quad (8)$$

$$n_\pm(t) = e^{\mp itA^{1/2}}n_{\pm 0} \pm i \int_0^t e^{\mp i(t-\tau)A^{1/2}}A^{-1/2}(|u(\tau)|^2)d\tau. \quad (9)$$

Conversely, if

$$(u, n_+, n_-) \in X^{s,b}[0, T] \times X_+^{\sigma,b}[0, T] \times X_-^{\sigma,b}[0, T]$$

is a solution of (5),(6) with data  $u(0) = u_0 \in H^s$  and  $n_\pm(0) = n_{\pm 0} \in H^\sigma$ , then we define  $n := \frac{1}{2}(n_+ + n_-)$ ,  $2iA^{-\frac{1}{2}}\partial_t n := n_+ - n_-$  and conclude that

$$(u, n, \partial_t n) \in X^{s,b}[0, T] \times (X_+^{\sigma,b}[0, T] + X_-^{\sigma,b}[0, T]) \times (X_+^{\sigma-1,b}[0, T] + X_-^{\sigma-1,b}[0, T])$$

is a solution of (1),(2) with data  $u(0) = u_0 \in H^s$  and

$$n(0) = n_0 = \frac{1}{2}(n_+(0) + n_-(0)) \in H^\sigma, \quad \partial_t n(0) = \frac{1}{2i}A^{\frac{1}{2}}(n_+(0) - n_-(0)) \in H^{\sigma-1}.$$

If  $(u, n_+, n_-) \in C^0([0, T], H^s) \times C^0([0, T], H^\sigma) \times C^0([0, T], H^\sigma)$ , then we also have  $(u, n, \partial_t n) \in C^0([0, T], H^s) \times C^0([0, T], H^\sigma) \times C^0([0, T], H^{\sigma-1})$ .

Our local well-posedness result reads as follow:

**Theorem 1.1.** *The Cauchy problem for the Klein - Gordon - Schrödinger system (1),(2),(3) is locally well-posed for data*

$$u_0 \in H^s(\mathbb{R}^3), \quad n_0 \in H^\sigma(\mathbb{R}^3), \quad n_1 \in H^{\sigma-1}(\mathbb{R}^3)$$

under the assumptions

$$s > -\frac{1}{4}, \quad \sigma > -\frac{1}{2}, \quad \sigma - 2s < \frac{3}{2}, \quad \sigma - 2 < s < \sigma + 1.$$

More precisley, there exists  $T > 0$ ,  $T = T(\|u_0\|_{H^s}, \|n_0\|_{H^\sigma}, \|n_1\|_{H^{\sigma-1}})$  and a unique solution

$$u \in X^{s, \frac{1}{2}+}[0, T],$$

$$n \in X_+^{\sigma, \frac{1}{2}+}[0, T] + X_-^{\sigma, \frac{1}{2}+}[0, T], \quad \partial_t n \in X_+^{\sigma-1, \frac{1}{2}+}[0, T] + X_-^{\sigma-1, \frac{1}{2}+}[0, T].$$

This solution has the property

$$u \in C^0([0, T], H^s(\mathbb{R}^3)), \quad n \in C^0([0, T], H^\sigma(\mathbb{R}^3)), \quad \partial_t n \in C^0([0, T], H^{\sigma-1}(\mathbb{R}^3)).$$

These conditions are sharp up to the endpoints, more precisely we get

**Theorem 1.2.** *Let  $u_0 \in H^s(\mathbb{R}^3)$ ,  $n_0 \in H^\sigma(\mathbb{R}^3)$ ,  $n_1 \in H^{\sigma-1}(\mathbb{R}^3)$ . Then the flow map  $(u_0, n_0, n_1) \mapsto (u(t), n(t), \partial_t n(t))$ ,  $t \in [0, T]$ , does not belong to  $C^2$  for any  $T > 0$ , provided  $\sigma - 2s - \frac{3}{2} > 0$  or  $s < -\frac{1}{4}$  or  $\sigma < -\frac{1}{2}$ .*

The necessary estimates for the nonlinearities required in the local existence results are false if the assumptions regarding the smoothness of the data are violated. This is proven in section 4, Prop. 4.1 and Prop. 4.2.

The unconditional uniqueness result is the following:

**Theorem 1.3.** *Let  $u_0 \in L^2(\mathbb{R}^3)$ ,  $n_0 \in L^2(\mathbb{R}^3)$ ,  $n_1 \in H^{-1}(\mathbb{R}^3)$  be given. The Klein - Gordon - Schrödinger system (1),(2),(3) is unconditionally globally well-posed, i.e. there exists a solution unique in*

$$(u, n, \partial_t n) \in C^0(\mathbb{R}^+, L^2(\mathbb{R}^3)) \times C^0(\mathbb{R}^+, L^2(\mathbb{R}^3)) \times C^0(\mathbb{R}^+, H^{-1}(\mathbb{R}^3)).$$

We use the following notation. The Fourier transform is denoted by  $\hat{\cdot}$  or  $\mathcal{F}$  and its inverse by  $\check{\cdot}$  or  $\mathcal{F}^{-1}$ , where it should be clear from the context, whether it is taken with respect to the space and time variables simultaneously or only with respect to the space variables. For real numbers  $a$  we denote by  $a+$  and  $a-$  a number sufficiently close to  $a$ , but larger and smaller than  $a$ , respectively.

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## 2. (CONDITIONAL) LOCAL WELL-POSEDNESS

**Theorem 2.1.** *Assume  $\frac{1}{4} < b_1 \leq \frac{1}{2}$ ,  $b, b_2 \geq \frac{1}{2}$ ,  $s > -1$ ,*

$$\sigma > \frac{1}{2} - 2b_1, \quad (10)$$

$$s + \sigma > -2b_2, \quad (11)$$

$$s < \sigma + 2b_1. \quad (12)$$

*If  $0 < T \leq 1$  and  $u \in X^{s, b_2}[0, T]$ ,  $v \in X_{\pm}^{\sigma, b}[0, T]$  we have*

$$\|uv\|_{X^{s, -b_1}[0, T]} \leq c \|u\|_{X^{s, b_2}[0, T]} \|v\|_{X_{\pm}^{\sigma, b}[0, T]}.$$

**Remark:** A possible choice of the parameters is:

$$b_1 = \frac{1}{2} -, \quad b = b_2 = \frac{1}{2}, \quad \sigma > -\frac{1}{2}, \quad s > -\frac{1}{2}, \quad s < \sigma + 1.$$

Because we are going to use dyadic decompositions of  $\hat{u}$  and  $\hat{v}$  we take the notation from [4] and start by choosing a function  $\psi \in C_0^\infty((-2, 2))$ , which is even and nonnegative with  $\psi(r) = 1$  for  $|r| \leq 1$ . Defining  $\psi_N(r) = \psi(\frac{r}{N}) - \psi(\frac{2r}{N})$  for dyadic numbers  $N = 2^n \geq 2$  and  $\psi_1 = \psi$  we have  $1 = \sum_{N \geq 1} \psi_N$ . Thus  $\text{supp } \psi_1 \subset [-2, 2]$  and  $\text{supp } \psi_N \subset [-2N, -N/2] \cup [N/2, 2N]$  for  $N \geq 2$ . For  $f : \mathbb{R}^3 \rightarrow \mathbb{C}$  we define the dyadic frequency localization operators  $P_N$  by

$$\mathcal{F}_x(P_N f)(\xi) = \psi_N(|\xi|) \mathcal{F}_x f(\xi).$$

For  $u : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{C}$  we define the modulation localization operators

$$\begin{aligned} \mathcal{F}(S_L u)(\tau, \xi) &= \psi_L(\tau + |\xi|^2) \mathcal{F}u(\tau, \xi) \\ \mathcal{F}(W_L^\pm u)(\tau, \xi) &= \psi_L(\tau \pm |\xi|) \mathcal{F}u(\tau, \xi) \end{aligned}$$

in the Schrödinger case and the wave case.

*Proof of Theorem 2.1.* Defining

$$I(f, g_1, g_2) = \int f(\xi_1 - \xi_2, \tau_1 - \tau_2) g_1(\xi_1, \tau_1) g_2(\xi_2, \tau_2) d\xi_1 d\xi_2 d\tau_1 d\tau_2$$

we have to show

$$|I(\hat{v}, \hat{u}_1, \hat{u}_2)| \lesssim \|u_1\|_{X^{-s, b_1}} \|u_2\|_{X^{s, b_2}} \|v\|_{X_{\pm}^{\sigma, b}}.$$

We use dyadic decompositions

$$u_k = \sum_{N_k, L_k \geq 1} S_{L_k} P_{N_k} u_k, \quad v = \sum_{N, L \geq 1} W_L^\pm P_N v.$$

Defining

$$g_k^{L_k, N_k} = \mathcal{F} S_{L_k} P_{N_k} u_k, \quad f^{L, N} = \mathcal{F} W_L^\pm P_N v$$

we have

$$I(\widehat{v}, \widehat{u}_1, \widehat{u}_2) = \sum_{N, N_1, N_2 \geq 1, L, L_1, L_2 \geq 1} I(f^{L, N}, g_1^{L_1, N_1}, g_2^{L_2, N_2}).$$

**Case 1:**  $N_1 \sim N_2 \gtrsim N \gg 1$

**a.** In the case  $L, L_1, L_2 \leq N_1^2$  we get by [4, formula (3.24)]:

$$\begin{aligned} & |I(f^{L, N}, g_1^{L_1, N_1}, g_2^{L_2, N_2})| \\ & \lesssim N_1^{-\frac{1}{2}+} L^{\frac{1}{2}} \|f^{L, N}\|_{L^2} L_1^{\frac{1}{2}} \|g_1^{L_1, N_1}\|_{L^2} L_2^{\frac{1}{2}} \|g_2^{L_2, N_2}\|_{L^2} \\ & \lesssim A N_1^{-\frac{1}{2}+s+} N_2^{-s+} N^{-\sigma+} L_1^{\frac{1}{2}-b_1+} L_2^{\frac{1}{2}-b_2+} L^{\frac{1}{2}-b+} \\ & \lesssim A N_1^{\frac{1}{2}-2b_1+} N^{-\sigma+} \\ & \lesssim A, \end{aligned}$$

because  $b_1 > \frac{1}{4}$  and (10). Here

$$A := N^{\sigma-} L^{b-} \|f^{L, N}\|_{L^2} N_1^{-s-} L_1^{b_1-} \|g_1^{L_1, N_1}\|_{L^2} N_2^{s-} L_2^{b_2-} \|g_2^{L_2, N_2}\|_{L^2}.$$

Dyadic summation over  $L, L_1, L_2 \geq 1$  and  $N, N_1, N_2$  gives

$$\begin{aligned} & \sum_{1 \ll N \lesssim N_1 \sim N_2} \sum_{1 \leq L, L_1, L_2 \leq N_1^2} |I(f^{L, N}, g_1^{L_1, N_1}, g_2^{L_2, N_2})| \\ & \lesssim \left( \sum_{N, L} (N^\sigma L^b \|f^{L, N}\|_{L^2})^2 \right)^{\frac{1}{2}} \left( \sum_{N_1, L_1} (N_1^{-s} L_1^{b_1} \|g_1^{L_1, N_1}\|_{L^2})^2 \right)^{\frac{1}{2}} \\ & \quad \left( \sum_{N_2, L_2} (N_2^s L_2^{b_2} \|g_2^{L_2, N_2}\|_{L^2})^2 \right)^{\frac{1}{2}} \\ & \lesssim \|v\|_{X_{\pm}^{\sigma, b}} \|u_1\|_{X^{-s, b_1}} \|u_2\|_{X^{s, b_2}} \end{aligned}$$

by almost orthogonality.

**b.** Similarly in the case  $N_1^2 < \max(L, L_1, L_2)$  we get by [4, formula (3.25)]:

$$\begin{aligned} & |I(f^{L, N}, g_1^{L_1, N_1}, g_2^{L_2, N_2})| \\ & \lesssim N_1^{-\frac{1}{2}} \frac{N_1}{\max(L, L_1, L_2)^{\frac{1}{2}}} L^{\frac{1}{2}} \|f^{L, N}\|_{L^2} L_1^{\frac{1}{2}} \|g_1^{L_1, N_1}\|_{L^2} L_2^{\frac{1}{2}} \|g_2^{L_2, N_2}\|_{L^2} \\ & \lesssim A N_1^{-\frac{1}{2}+s+} N_2^{-s+} N^{-\sigma+} L_1^{\frac{1}{2}-b_1+} L_2^{\frac{1}{2}-b_2+} L^{\frac{1}{2}-b+} \frac{N_1}{\max(L, L_1, L_2)^{\frac{1}{2}}} \\ & \lesssim A N_1^{\frac{1}{2}+} N^{-\sigma+} \frac{1}{\max(L, L_1, L_2)^{b_1-}} \\ & \lesssim A N_1^{\frac{1}{2}-2b_1+} N^{-\sigma+} \\ & \lesssim A \end{aligned}$$

as in case a. Dyadic summation gives the claimed estimate.

**Case 2:**  $N_1 \ll N_2$  ( $\implies N \sim N_2$ )

**a.** In the case  $L_2 \ll N_2^2$  we get by [4, formula (3.26) and (3.28)] :  
 $\max(L, L_1, L_2) \gtrsim N_2^2$  and

$$\begin{aligned}
& |I(f^{L,N}, g_1^{L_1, N_1}, g_2^{L_2, N_2})| \\
& \lesssim N_1 N_2^{-\frac{1}{2}} (L_1 L_2 L)^{\frac{1}{2}} \max(L, L_1, L_2)^{-\frac{1}{2}} \|f^{L,N}\|_{L^2} \|g_1^{L_1, N_1}\|_{L^2} \|g_2^{L_2, N_2}\|_{L^2} \\
& \lesssim A N_1^{1+s+} N_2^{-\frac{1}{2}-s+} N^{-\sigma+} L_1^{\frac{1}{2}-b_1+} L_2^{\frac{1}{2}-b_2+} L^{\frac{1}{2}-b+} \max(L, L_1, L_2)^{-\frac{1}{2}} \\
& \lesssim A N_2^{\frac{1}{2}-\sigma+} \max(L, L_1, L_2)^{-b_1+} \\
& \lesssim A N_2^{\frac{1}{2}-2b_1-\sigma+} \\
& \lesssim A
\end{aligned}$$

using  $s > -1$  and (10). Dyadic summation gives the desired bound as in case 1.

**b.** In the case  $L_2 \gtrsim N_2^2$  we consider 3 subcases using the proof of [4, Prop. 3.8]:

**b1.**  $L \leq L_1$  and  $N_1^2 \leq \max(L, L_1) = L_1$ .

$$\begin{aligned}
|I(f^{L,N}, g_1^{L_1, N_1}, g_2^{L_2, N_2})| & \lesssim N_1^{\frac{3}{2}} L^{\frac{1}{2}} \|f^{L,N}\|_{L^2} \|g_1^{L_1, N_1}\|_{L^2} \|g_2^{L_2, N_2}\|_{L^2} \\
& \lesssim A N_1^{\frac{3}{2}+s+} N_2^{-s+} N^{-\sigma+} L_1^{-b_1+} L_2^{-b_2+} L^{\frac{1}{2}-b+} \\
& \lesssim A N_1^{\frac{3}{2}+s+} N_2^{-s-\sigma+} L_1^{-b_1+} L_2^{-b_2+} \\
& \lesssim A N_1^{\frac{3}{2}+s-2b_1+} N_2^{-s-\sigma-2b_2+} \\
& \lesssim A N_1^{\frac{3}{2}-\sigma-2b_1-2b_2+} \\
& \lesssim A
\end{aligned}$$

using (11),(10) and  $b_2 \geq \frac{1}{2}$ . Dyadic summation gives the claimed estimate.

**b2.**  $L_1 < L$  and  $N_1^2 \leq \max(L, L_1) = L$ .

We have

$$\begin{aligned}
|I(f^{L,N}, g_1^{L_1, N_1}, g_2^{L_2, N_2})| & \lesssim N_1^{\frac{3}{2}} L_1^{\frac{1}{2}} \|f^{L,N}\|_{L^2} \|g_1^{L_1, N_1}\|_{L^2} \|g_2^{L_2, N_2}\|_{L^2} \\
& \lesssim A N_1^{\frac{3}{2}+s+} N_2^{-s+} N^{-\sigma+} L_1^{\frac{1}{2}-b_1+} L_2^{-b_2+} L^{-b+} \\
& \lesssim A N_1^{\frac{3}{2}+s+} N_2^{-s-\sigma+} L^{\frac{1}{2}-b_1-b+} L_2^{-b_2+} \\
& \lesssim A N_1^{\frac{3}{2}+s+} N_2^{-s-\sigma+} N_1^{1-2b_1-2b+} N_2^{-2b_2+} \\
& \lesssim A N_1^{\frac{3}{2}+s+1-2b_1-2b+} N_2^{-s-\sigma-2b_2+} \\
& \lesssim A N_1^{\frac{5}{2}-\sigma-2b_1-2b_2-2b+} \\
& \lesssim A
\end{aligned}$$

again using (11),(10) and  $b, b_2 \geq \frac{1}{2}$ .

**b3.**  $N_1^2 > \max(L, L_1)$ .

We have

$$\begin{aligned}
|I(f^{L,N}, g_1^{L_1, N_1}, g_2^{L_2, N_2})| & \lesssim N_1^{\frac{1}{2}} L^{\frac{1}{2}} L_1^{\frac{1}{2}} \|f^{L,N}\|_{L^2} \|g_1^{L_1, N_1}\|_{L^2} \|g_2^{L_2, N_2}\|_{L^2} \\
& \lesssim A N_1^{\frac{1}{2}+s+} N_2^{-s+} N^{-\sigma+} L_1^{\frac{1}{2}-b_1+} L_2^{-b_2+} L^{\frac{1}{2}-b+} \\
& \lesssim A N_1^{\frac{1}{2}+s+} N_2^{-s-\sigma+} N_1^{1-2b_1+} N_2^{-2b_2+} \\
& \lesssim A N_1^{\frac{3}{2}-\sigma-2b_1-2b_2+} \\
& \lesssim A
\end{aligned}$$

again using (11),(10) and  $b_2 \geq \frac{1}{2}$ .

**Case 3:**  $N_2 \ll N_1$  ( $\implies N \sim N_1$ ).

Interchanging the roles of  $N_1$  and  $N_2$  as well as  $L_1$  and  $L_2$  we consider different cases.

**a.**  $L_1 \ll N_1^2$ .

We use [4, formula (3.26) and (3.28)] and get  $\max(L, L_1, L_2) \gtrsim N_1^2$  and

$$\begin{aligned}
& |I(f^{L,N}, g_1^{L_1, N_1}, g_2^{L_2, N_2})| \\
& \lesssim N_2 N_1^{-\frac{1}{2}} (L_1 L_2 L)^{\frac{1}{2}} \max(L, L_1, L_2)^{-\frac{1}{2}} \|f^{L,N}\|_{L^2} \|g_1^{L_1, N_1}\|_{L^2} \|g_2^{L_2, N_2}\|_{L^2} \\
& \lesssim A N_1^{s-\frac{1}{2}+} N_2^{-s+1+} N^{-\sigma+} L_1^{\frac{1}{2}-b_1+} L_2^{\frac{1}{2}-b_2+} L^{\frac{1}{2}-b+} \max(L, L_1, L_2)^{-\frac{1}{2}} \\
& \lesssim A N_1^{s-\frac{1}{2}-\sigma+} N_2^{-s+1+} \max(L, L_1, L_2)^{-b_1+} \\
& \lesssim A N_1^{s-\frac{1}{2}-\sigma-2b_1+} N_2^{-s+1+}.
\end{aligned}$$

If  $s \leq 1$  we get the bound  $A N_1^{\frac{1}{2}-\sigma-2b_1} \lesssim A$  by (10), whereas for  $s > 1$  we estimate by  $A N_1^{s-\frac{1}{2}-\sigma-2b_1+} \lesssim A N_1^{-\frac{1}{2}} \lesssim A$  using (12). Dyadic summation gives the claimed estimate.

**b.** In the case  $L_1 \gtrsim N_1^2$  we consider 3 subcases using the proof of [4, Prop. 3.8]

**b1.**  $L \leq L_2$ ,  $N_2^2 \leq \max(L, L_2) = L_2$ .

We get

$$\begin{aligned}
|I(f^{L,N}, g_1^{L_1, N_1}, g_2^{L_2, N_2})| & \lesssim N_2^{\frac{3}{2}} L^{\frac{1}{2}} \|f^{L,N}\|_{L^2} \|g_1^{L_1, N_1}\|_{L^2} \|g_2^{L_2, N_2}\|_{L^2} \\
& \lesssim A N_1^{s+} N_2^{-s+\frac{3}{2}+} N^{-\sigma+} L_1^{-b_1+} L_2^{-b_2+} L^{\frac{1}{2}-b+} \\
& \lesssim A N_1^{s-\sigma-2b_1+} N_2^{-s+\frac{3}{2}-2b_2+}.
\end{aligned}$$

If  $s < \frac{3}{2} - 2b_2$  we get the bound  $A N_1^{\frac{3}{2}-\sigma-2b_1-2b_2+} \lesssim A N_1^{1-2b_2} \lesssim A$  by (10) and  $b_2 \geq \frac{1}{2}$ , whereas in the case  $s \geq \frac{3}{2} - 2b_2$  we estimate by  $A N_1^{s-\sigma-2b_1+} \lesssim A$  using (12).

**b2.**  $L_2 < L$ ,  $N_2^2 \leq \max(L, L_2) = L$ .

We get

$$\begin{aligned}
|I(f^{L,N}, g_1^{L_1, N_1}, g_2^{L_2, N_2})| & \lesssim N_2^{\frac{3}{2}} L^{\frac{1}{2}} \|f^{L,N}\|_{L^2} \|g_1^{L_1, N_1}\|_{L^2} \|g_2^{L_2, N_2}\|_{L^2} \\
& \lesssim A N_1^{s+} N_2^{-s+\frac{3}{2}+} N^{-\sigma+} L_1^{-b_1+} L_2^{\frac{1}{2}-b_2+} L^{-b+} \\
& \lesssim A N_1^{s-\sigma-2b_1+} N_2^{-s+\frac{3}{2}-2b+}.
\end{aligned}$$

If  $s < \frac{3}{2} - 2b$  we get the bound  $A N_1^{\frac{3}{2}-\sigma-2b_1-2b+} \lesssim A N_1^{1-2b} \lesssim A$  by (10) and  $b \geq \frac{1}{2}$ , whereas in the case  $s \geq \frac{3}{2} - 2b$  we estimate by  $A N_1^{s-\sigma-2b_1+} \lesssim A$  using (12).

**b3.**  $N_2^2 > \max(L, L_2)$ .

We get

$$\begin{aligned}
|I(f^{L,N}, g_1^{L_1, N_1}, g_2^{L_2, N_2})| & \lesssim N_2^{\frac{1}{2}} L^{\frac{1}{2}} \|f^{L,N}\|_{L^2} \|g_1^{L_1, N_1}\|_{L^2} \|g_2^{L_2, N_2}\|_{L^2} \\
& \lesssim A N_1^{s+} N_2^{-s+\frac{1}{2}+} N^{-\sigma+} L_1^{-b_1+} L_2^{\frac{1}{2}-b_2+} L^{\frac{1}{2}-b+} \\
& \lesssim A N_1^{s-\sigma-2b_1+} N_2^{-s+\frac{1}{2}+}.
\end{aligned}$$

If  $s \leq \frac{1}{2}$  we get the bound  $A N_1^{\frac{1}{2}-\sigma-2b_1+} \lesssim A$  by (10), whereas in the case  $s > \frac{1}{2}$  we estimate by  $A N_1^{s-\sigma-2b_1+} \lesssim A$  using (12), which gives the desired bound after dyadic summation.

**Case 4:**  $N \lesssim 1$

In this case we need no dyadic decomposition. We estimate directly using  $\langle \xi_1 \rangle \sim$

$\langle \xi_2 \rangle$  and  $\xi = \xi_1 - \xi_2$ ,  $\tau = \tau_1 - \tau_2$  :

$$\begin{aligned}
& \int_{\langle \xi \rangle \lesssim 1, \langle \xi_1 \rangle \sim \langle \xi_2 \rangle} \widehat{v}(\xi, \tau) \widehat{u}_1(\xi_1, \tau_1) \widehat{u}_2(\xi_2, \tau_2) d\xi_1 d\xi_2 d\tau_1 d\tau_2 \\
& \lesssim \int_{\langle \xi \rangle \lesssim 1, \langle \xi_1 \rangle \sim \langle \xi_2 \rangle} \frac{\widehat{v}(\xi, \tau)}{\langle \xi \rangle^{\frac{3}{2}-\sigma+}} \frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{-s}} \frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle \xi_2 \rangle^s} d\xi_1 d\xi_2 d\tau_1 d\tau_2 \\
& \lesssim \|\mathcal{F}^{-1}(\frac{\widehat{v}(\xi, \tau)}{\langle \xi \rangle^{\frac{3}{2}-\sigma+}})\|_{L_t^3 L_x^\infty} \|\mathcal{F}^{-1}(\frac{\widehat{u}_1(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{-s}})\|_{L_t^3 L_x^2} \|\mathcal{F}^{-1}(\frac{\widehat{u}_2(\xi_2, \tau_2)}{\langle \xi_2 \rangle^s})\|_{L_t^3 L_x^2} \\
& \lesssim \|\mathcal{F}^{-1}(\frac{\widehat{v}(\xi, \tau)}{\langle \xi \rangle^{-\sigma}})\|_{L_t^3 L_x^2} \|u_1\|_{X^{-s, \frac{1}{6}+}} \|u_2\|_{X^{s, \frac{1}{6}+}} \\
& \lesssim \|v\|_{X_{\pm}^{\sigma, \frac{1}{6}+}} \|u_1\|_{X^{-s, \frac{1}{6}+}} \|u_2\|_{X^{s, \frac{1}{6}+}} .
\end{aligned}$$

by Sobolev's embedding theorem. This is more than enough for our claimed estimate and completes the proof of Theorem 2.1.  $\square$

**Theorem 2.2.** Assume  $s > -\frac{1}{4}$ ,  $\sigma - 2s < \frac{3}{2}$ ,  $\sigma - 2 < s$ . If  $0 < T \leq 1$  and  $u_1, u_2 \in X^{s, \frac{1}{2}-}[0, T]$  the following estimate holds:

$$\|A^{-1/2}(u_1 \overline{u}_2)\|_{X_{\pm}^{\sigma, -\frac{1}{2}+}[0, T]} \leq c \|u_1\|_{X^{s, \frac{1}{2}-}[0, T]} \|u_2\|_{X^{s, \frac{1}{2}-}[0, T]} .$$

*Proof.* We have to show

$$|I(\widehat{v}, \widehat{u}_1, \widehat{u}_2)| \lesssim \|u_1\|_{X^{s, \frac{1}{2}-}} \|u_2\|_{X^{s, \frac{1}{2}-}} \|v\|_{X_{\pm}^{1-\sigma, \frac{1}{2}-}} .$$

Using dyadic decompositions as in the proof of Theorem 2.1 we consider different cases.

**Case 1:**  $N_1 \sim N_2 \gtrsim N \gg 1$

**a.** In the case  $L, L_1, L_2 \leq N_1^2$  we get by [4, formula (3.24)]

$$\begin{aligned}
& |I(f^{L, N}, g_1^{L_1, N_1}, g_2^{L_2, N_2})| \\
& \lesssim N_1^{-\frac{1}{2}+} L^{\frac{1}{2}} \|f^{L, N}\|_{L^2} L_1^{\frac{1}{2}} \|g_1^{L_1, N_1}\|_{L^2} L_2^{\frac{1}{2}} \|g_2^{L_2, N_2}\|_{L^2} \\
& \lesssim B N_1^{-\frac{1}{2}-s+} N_2^{-s+} N^{\sigma-1+} L_1^{0+} L_2^{0+} L^{0+} \\
& \lesssim B N_1^{-\frac{1}{2}-2s+} N^{\sigma-1+} ,
\end{aligned}$$

where

$$B := N^{1-\sigma-} L^{\frac{1}{2}-} \|f^{L, N}\|_{L^2} N_1^{s-} L_1^{\frac{1}{2}-} \|g_1^{L_1, N_1}\|_{L^2} N_2^{s-} L_2^{\frac{1}{2}-} \|g_2^{L_2, N_2}\|_{L^2} .$$

If  $\sigma \leq 1$  we get the bound  $\lesssim B N_1^{-\frac{1}{2}-2s+} \lesssim B$ , because  $s > -\frac{1}{4}$ , whereas in the case  $\sigma > 1$  we estimate by  $\lesssim B N_1^{-\frac{3}{2}-2s+\sigma+} \lesssim B$ , because  $\sigma - 2s < \frac{3}{2}$ . Dyadic summation over  $L, L_1, L_2 \geq 1$  and  $N, N_1, N_2$  gives the claimed result.

**b.** In the case  $N_1^2 < \max(L, L_1, L_2)$  we get by [4, formula (3.25)] the same estimate as in a. with  $N_1^{-\frac{1}{2}+}$  replaced by  $N_1^{-\frac{1}{2}} N_1 \max(L, L_1, L_2)^{-\frac{1}{2}} \leq N_1^{-\frac{1}{2}}$ , so that the same argument applies.

**Case 2:**  $N_1 \ll N_2 (\implies N \sim N_2)$  (or  $N_2 \ll N_1$ , which is the same problem).

**a.** In the case  $L_2 \gtrsim N_2^2$  we get by [4, formula (3.27)] :

$$\begin{aligned}
& |I(f^{L, N}, g_1^{L_1, N_1}, g_2^{L_2, N_2})| \\
& \lesssim N_1^{\frac{1}{2}} \min(L, L_1)^{\frac{1}{2}} \min(N_1^2, \max(L, L_1))^{\frac{1}{2}} \|f^{L, N}\|_{L^2} \|g_1^{L_1, N_1}\|_{L^2} \|g_2^{L_2, N_2}\|_{L^2} \\
& \lesssim B N_1^{\frac{1}{2}-s+} N_2^{-s+} N^{\sigma-1+} L^{-\frac{1}{2}+} L_1^{-\frac{1}{2}+} L_2^{-\frac{1}{2}+} \min(L, L_1)^{\frac{1}{2}} \min(N_1^2, \max(L, L_1))^{\frac{1}{2}}
\end{aligned}$$



**a1.**  $L, L_1 \leq N_1^2$

In this case we can estimate this by

$$\lesssim BN_1^{\frac{1}{2}-s+} N_2^{-s+} N^{\sigma-1+} L_2^{-\frac{1}{2}+} \lesssim BN_1^{\frac{1}{2}-s+} N^{\sigma-2-s+}.$$

If  $s \leq \frac{1}{2}$  we get the bound  $\lesssim BN^{-\frac{3}{2}-2s+\sigma+} \lesssim B$ , because  $\sigma - 2s < \frac{3}{2}$ , whereas in the case  $s > \frac{1}{2}$  we estimate by  $\lesssim BN^{\sigma-2-s+} \lesssim B$ , because  $\sigma - 2 < s$ . Dyadic summation gives the desired bound.

**a2.**  $\max(L, L_1) > N_1^2$

We estimate in this case by

$$\lesssim BN_1^{\frac{3}{2}-s+} N_2^{-s+} N^{\sigma-1+} L_2^{-\frac{1}{2}+} \max(L, L_1)^{-\frac{1}{2}+} \lesssim BN_1^{\frac{1}{2}-s+} N^{\sigma-2-s+},$$

the same bound as in a1.

**b.** In the case  $L_2 \ll N_2^2$  we get by [4, formula (3.26) and (3.28)] :  $\max(L, L_1, L_2) \gtrsim N_2^2$  and

$$\begin{aligned} & |I(f^{L,N}, g_1^{L_1, N_1}, g_2^{L_2, N_2})| \\ & \lesssim N_1 N_2^{-\frac{1}{2}} (L_1 L_2 L)^{\frac{1}{2}} \max(L, L_1, L_2)^{-\frac{1}{2}} \|f^{L,N}\|_{L^2} \|g_1^{L_1, N_1}\|_{L^2} \|g_2^{L_2, N_2}\|_{L^2} \\ & \lesssim BN_1^{1-s+} N_2^{-\frac{1}{2}-s+} N^{\sigma-1+} (L_1 L_2 L)^{0+} \max(L; L_1, L_2)^{-\frac{1}{2}} \\ & \lesssim BN_1^{1-s+} N^{-\frac{1}{2}-s+\sigma-2+} \\ & \lesssim BN_1^{\frac{1}{2}-s+} N^{\sigma-2-s+}, \end{aligned}$$

the same bound as in a1.

**Case 3:**  $N \lesssim 1$  ( $\implies N_1 \sim N_2$  or  $N, N_1, N_2 \lesssim 1$ .)

Assuming without loss of generality  $L_1 \leq L_2$  and using the bilinear Strichartz type estimate [4, Prop. 4.3] we get

$$\begin{aligned} & |I(f^{L,N}, g_1^{L_1, N_1}, g_2^{L_2, N_2})| \\ & \leq \|f^{L,N} g_1^{L_1, N_1}\|_{L^2} \|g_2^{L_2, N_2}\|_{L^2} \\ & \lesssim \min(N, N_1) N_1^{-\frac{1}{2}} L^{\frac{1}{2}} L_1^{\frac{1}{2}} \|f^{L,N}\|_{L^2} \|g_1^{L_1, N_1}\|_{L^2} \|g_2^{L_2, N_2}\|_{L^2} \\ & \lesssim N_1^{-\frac{1}{2}} L^{\frac{1}{2}} \|f^{L,N}\|_{L^2} L_1^{\frac{1}{4}} \|g_1^{L_1, N_1}\|_{L^2} L_2^{\frac{1}{4}} \|g_2^{L_2, N_2}\|_{L^2}. \end{aligned}$$

Furthermore we get by [4, formula (4.22)]

$$|I(f^{L,N}, g_1^{L_1, N_1}, g_2^{L_2, N_2})| \lesssim L^{\frac{1}{3}} \|f^{L,N}\|_{L^2} L_1^{\frac{1}{3}} \|g_1^{L_1, N_1}\|_{L^2} L_2^{\frac{1}{3}} \|g_2^{L_2, N_2}\|_{L^2},$$

so that by interpolation we arrive at

$$\begin{aligned} |I(f^{L,N}, g_1^{L_1, N_1}, g_2^{L_2, N_2})| & \lesssim N_1^{-\frac{1}{2}+} L^{\frac{1}{2}-} \|f^{L,N}\|_{L^2} L_1^{\frac{1}{4}+} \|g_1^{L_1, N_1}\|_{L^2} L_2^{\frac{1}{4}+} \|g_2^{L_2, N_2}\|_{L^2} \\ & \lesssim BN_1^{-\frac{1}{2}-s+} N_2^{-s+} N^{\sigma-1+} \\ & \lesssim BN_1^{-\frac{1}{4}-s+} N_2^{-\frac{1}{4}-s+} \\ & \lesssim B \end{aligned}$$

using  $s > -\frac{1}{4}$ . Dyadic summation in all cases completes the proof of Theorem 2.2.  $\square$

*Proof of Theorem 1.1.* It is by now standard to use (the remark to) Theorem 2.1 and Theorem 2.2 to show the local well-posedness result (Theorem 1.1) for the system (5),(6),(7) as an application of the contraction mapping principle. For details of the method we refer to [7]. This solution then immediately leads to a solution of the Klein-Gordon-Schrödinger system (1),(2),(3) with the required properties as explained before Theorem 1.1.

Moreover, if  $(u, n, \partial_t n)$  is a solution of (the system of integral equations belonging to) (1),(1),(3) with  $u \in X^{s, \frac{1}{2}+}[0, T]$  and data  $u_0 \in H^s$ ,  $n_0 \in H^\sigma$ ,  $n_1 \in H^\sigma$ , then  $n_\pm$  defined by (9) belongs to  $X_\pm^{\sigma, \frac{1}{2}+}[0, T]$  by Theorem 2.2 and thus  $n = \frac{1}{2}(n_+ + n_-)$  belongs to  $X_+^{\sigma, \frac{1}{2}+}[0, T] + X_-^{\sigma, \frac{1}{2}+}[0, T]$  and  $\partial_t n = \frac{1}{2i}A^{\frac{1}{2}}(n_+ - n_-)$  belongs to  $X_+^{\sigma-1, \frac{1}{2}+}[0, T] + X_-^{\sigma-1, \frac{1}{2}+}[0, T]$ , and one easily checks that  $(u, n_+, n_-)$  is a solution of the system (of integral equations belonging to) (5),(6),(7). But because this solution is uniquely determined the solution of the Klein - Gordon - Schrödinger system is also unique.  $\square$

### 3. UNCONDITIONAL UNIQUENESS

In this section we show that solutions of the KGS system are unique in its natural solution spaces in the important case, where the Cauchy data for the Schrödinger part belong to  $L^2$  and the data for the Klein-Gordon-part belong to  $L^2 \times H^{-1}$ , namely in the space  $C^0([0, T], L^2)$  and  $C^0([0, T], L^2) \times C^0([0, T], H^{-1})$ , respectively. This is of particular interest, because we know that in this case the solution exists globally in time by the result of [6].

First we show

**Proposition 3.1.** *Let  $u_0 \in L^2(\mathbb{R}^3)$ ,  $n_0 \in L^2(\mathbb{R}^3)$ ,  $n_1 \in H^{-1}(\mathbb{R}^3)$  and  $T > 0$  be given. Any solution*

$$(u, n_+, n_-) \in C^0([0, T], L^2(\mathbb{R}^3)) \times C^0([0, T], L^2(\mathbb{R}^3)) \times C^0([0, T], L^2(\mathbb{R}^3))$$

*of the system (5),(6),(7) belongs to*

$$X^{0-, \frac{1}{2}+}[0, T] \times X_+^{-\frac{1}{4}-, \frac{1}{2}+}[0, T] \times X_-^{-\frac{1}{4}-, \frac{1}{2}+}[0, T].$$

*Proof of Theorem 1.3.* We again remark that any solution

$$(u, n, \partial_t n) \in C^0([0, T], L^2(\mathbb{R}^3)) \times C^0([0, T], L^2(\mathbb{R}^3)) \times C^0([0, T], H^{-1}(\mathbb{R}^3))$$

of the Klein-Gordon-Schrödinger system (1),(2),(3) leads to a corresponding solution of the system (5),(6),(7) with

$$(u, n_+, n_-) \in C^0([0, T], L^2(\mathbb{R}^3)) \times C^0([0, T], L^2(\mathbb{R}^3)) \times C^0([0, T], L^2(\mathbb{R}^3)).$$

Thus combining Prop. 3.1 with the local well-posedness result Theorem 1.1 and the global existence result of [6] we immediately get Theorem 1.3.  $\square$

*Proof of Prop. 3.1.* For the first part we use an idea of Y. Zhou [16],[17]. By Sobolev's embedding theorem we get

$$\begin{aligned} \|n_\pm u\|_{L^2((0, T), H^{-\frac{3}{2}-})} &\lesssim \|n_\pm u\|_{L^2((0, T), L^1)} \\ &\lesssim T^{\frac{1}{2}} \|n_\pm\|_{L^\infty((0, T), L^2)} \|u\|_{L^\infty((0, T), L^2)} < \infty. \end{aligned}$$

so that from (5) we have  $u \in X^{-\frac{3}{2}-, 1}[0, T]$ , because

$$\|(i\partial_t + \Delta)u\|_{L^2((0, T), H^{-\frac{3}{2}-})}^2 + \|u\|_{L^2((0, T), H^{-\frac{3}{2}-})}^2 \sim \|u\|_{X^{-\frac{3}{2}-, 1}[0, T]}^2 < \infty.$$

Similarly we get

$$\| |u|^2 \|_{L^2((0, T), H^{-\frac{3}{2}-})} \lesssim T^{\frac{1}{2}} \|u\|_{L^\infty((0, T), L^2)}^2 < \infty$$

and therefore  $A^{-1/2}(|u|^2) \in L^2((0, T), H^{-\frac{1}{2}-})$ . From (6) we conclude  $n_\pm \in X_\pm^{-\frac{1}{2}-, 1}[0, T]$ .

Interpolation with  $u \in X^{0, 0}[0, T]$  and  $n_\pm \in X_\pm^{0, 0}[0, T]$  gives  $u \in X^{-\frac{3}{2}\Theta-, \Theta}[0, T]$ , especially  $u \in X^{-\frac{3}{4}-\frac{3}{2}\epsilon-, \frac{1}{2}+\epsilon}[0, T]$ , and  $n_\pm \in X_\pm^{-\frac{1}{2}\Theta-, \Theta}[0, T]$ , especially  $n_\pm \in X_\pm^{-\frac{1}{4}-, \frac{1}{2}+}[0, T]$  for any  $0 \leq \Theta \leq 1$ .

We now improve the regularity of  $u$  keeping the regularity of  $n$  fixed. We suppose  $u \in X^{-\frac{3}{4}(\frac{4}{5})^n-, \frac{1}{2}+}$  and  $n_{\pm} \in X_{\pm}^{-\frac{1}{4}-, \frac{1}{2}+}$ , which is fulfilled for  $n = 0$ , and want to conclude  $u \in X^{-\frac{3}{4}(\frac{4}{5})^n-, \frac{5}{8}-}$  for all  $n \in \mathbb{N} \cup \{0\}$ . Assuming this for the moment we interpolate this result with  $u \in X^{0,0}$  with interpolation parameter  $\Theta = \frac{4}{5}+$  and conclude  $u \in X^{-\frac{3}{4}(\frac{4}{5})^{n+1}-, \frac{1}{2}+}$ , so that the iteration works and finally gives  $u \in X^{0-, \frac{1}{2}+}[0, T]$ . Recalling  $n_{\pm} \in X_{\pm}^{-\frac{1}{4}-, \frac{1}{2}+}[0, T]$  the uniqueness part of our local well-posedness result Theorem 1.1 gives the claimed result. So we are done if we prove

$$u \in X^{s, \frac{1}{2}+\epsilon}, n_{\pm} \in X_{\pm}^{-\frac{1}{4}-, \frac{1}{2}+} \implies u \in X^{s, \frac{5}{8}-2\epsilon} \quad (13)$$

for  $-\frac{3}{4} - \frac{3}{2}\epsilon- \leq s < 0$  and any  $\epsilon > 0$ .

This means that we have to show

$$\|un_{\pm}\|_{X^{s, -\frac{3}{8}-\epsilon}} \lesssim \|u\|_{X^{s, \frac{1}{2}+\epsilon}} \|n_{\pm}\|_{X_{\pm}^{-\frac{1}{4}-, \frac{1}{2}+}}.$$

This is a consequence of Theorem 2.1, where we choose the parameters as follows:

$$b_1 = \frac{3}{8} + \epsilon, b_2 = \frac{1}{2} + \epsilon, b = \frac{1}{2}, \sigma = -\frac{1}{4}-.$$

Then one easily checks that the conditions (10), (11) and (12) are satisfied, provided  $s \geq -\frac{3}{4} - \frac{3}{2}\epsilon-$ .

This completes the proof of Prop. 3.1, and thus Theorem 1.3 is also proven.  $\square$

#### 4. SHARPNESS OF THE WELL-POSEDNESS RESULT

In this section we show that the local well-posedness result is sharp up to the endpoints. First we construct counterexamples which show that the threshold on the parameters  $s$  and  $\sigma$  in Theorem 2.1 and Theorem 2.2 is essentially necessary. We follow the arguments of [8] and [3].

**Proposition 4.1.** *Assume  $s \in \mathbb{R}$ ,  $b', b_1, b_2 \geq 0$  and  $\sigma < -\frac{1}{2}$ . Then the inequality*

$$\|uv\|_{X^{s, -b'}} \lesssim \|v\|_{X_{\pm}^{\sigma, b_1}} \|u\|_{X^{s, b_2}}$$

*is false.*

*Proof.* [3, Prop. 6.1]. The two dimensional argument carries over to three dimensions.  $\square$

**Proposition 4.2.** (a) *Assume  $\sigma, s \in \mathbb{R}$ ,  $b', b_1, b_2 \geq 0$  and  $\sigma - 2s - \frac{3}{2} > 0$ . Then the inequality*

$$\|A^{-1/2}(u\bar{w})\|_{X_{\pm}^{\sigma, -b'}} \lesssim \|u\|_{X^{s, b_1}} \|w\|_{X^{s, b_2}}$$

*is false.*

(b) *The same holds true, if  $s < -\frac{1}{4}$ .*

*Proof.* (a) [3, Prop. 6.2]. Replace  $\sigma$  by  $\sigma + 2$  and use their two-dimensional argument.

(b) follows immediately from the following

**Lemma 4.1.** *Assume  $\sigma, s \in \mathbb{R}$  and  $b', b_1, b_2 \geq 0$ . For any  $N \gg 1$  there exist functions  $u_N$  and  $w_N$  and a constant  $c_0 > 0$  independent of  $N$  such that*

$$\frac{\|A^{-1/2}(u_N \bar{w}_N)\|_{X_{\pm}^{\sigma, -b'}}}{\|u_N\|_{X^{s, b_1}} \|w_N\|_{X^{s, b_2}}} \geq c_0 N^{-2s-\frac{1}{2}}.$$

*Proof.* Let  $\widehat{u}_N = \chi_E$  (= characteristic function of the set  $E$ ), where

$$E = \{(\xi_1, \xi_2, \xi_3, \tau) \in \mathbb{R}^4 : N - \frac{1}{N} \leq \xi_1 \leq N + \frac{1}{N}, -1 \leq \xi_2, \xi_3 \leq 1, \\ -N^2 - 1 \leq \tau \leq -N^2 + 1\},$$

Moreover let  $\widehat{w}_N = \chi_F$ , where

$$F = \{-N + \frac{3}{N} \leq \xi_1 \leq -N + \frac{5}{N}, -1 \leq \xi_2, \xi_3 \leq 1, \\ -(N - \frac{4}{N})^2 - 1 \leq \tau \leq -(N - \frac{4}{N})^2 + 1\},$$

so that  $\langle \tau + |\xi|^2 \rangle \sim 1$  and  $\langle \xi \rangle \sim N$  on  $E$  and  $F$ . This implies  $\widehat{u_N \overline{w_N}}(\xi, \tau) \gtrsim N^{-1}$  on a rectangle  $G$  centered around points  $\xi_1^0 = 0(\frac{1}{N})$ ,  $\xi_2^0 = \xi_3^0 = 0$ ,  $\tau^0 = 0(1)$  of width  $0(\frac{1}{N}), 0(1), 0(1)$  and  $0(1)$ , respectively, so that  $\langle \tau + |\xi|^2 \rangle \sim 1$  and  $\langle \xi \rangle \sim 1$  on  $G$ . Consequently

$$\|A^{-1/2}(u_N \overline{w_N})\|_{X_{\pm}^{\sigma, -b'}} \gtrsim (\int_G (\langle \xi \rangle^{\sigma-1} \widehat{u_N \overline{w_N}}(\xi, \tau) \langle |\xi|^2 + \tau \rangle^{-b'})^2 d\xi d\tau)^{\frac{1}{2}} \gtrsim N^{-\frac{3}{2}},$$

whereas

$$\|u_N\|_{X^{s, b_1}} + \|w_N\|_{X^{s, b_2}} \lesssim N^{s-\frac{1}{2}}.$$

so that the proof is complete.  $\square$

The following Propositions 4.3, 4.4 and 4.5 show that the flow map of our Cauchy problem is not  $C^2$  so that the problem is ill-posed in this sense. This strategy of proof goes back to Bourgain [3], Tzvetkov [15] and Molinet-Saut-Tzvetkov [11],[12] and is taken up by Holmer [8]. Thus the proof of Theorem 1.2 immediately follows from these propositions by the arguments of Holmer [8].

**Proposition 4.3.** *Let  $0 < T \leq 1$ . For any  $N \gg 1$  there exists  $u_N \in H^s(\mathbb{R}^3)$  and a constant  $c_0 > 0$ , which is independent of  $N$  such that*

$$\sup_{|t| \leq T} \left\| \int_0^t e^{-i(t-t')A^{1/2}} A^{-1/2} (e^{it'\Delta} u_N \overline{e^{it'\Delta} u_N}) dt' \right\|_{H^s(\mathbb{R}^3)} \geq c_0 N^{\sigma-2s-\frac{3}{2}} \|u_N\|_{H^s(\mathbb{R}^3)}^2.$$

*Proof.* Let  $\widehat{u}_N := \chi_{D_1} + \chi_{D_2}$ , where

$$D_1 := \{\xi \in \mathbb{R}^3 : N+1 - N^{-1} \leq \xi_1 \leq N+1 + N^{-1}, -1 \leq \xi_2, \xi_3 \leq 1\}, \\ D_2 := \{\xi \in \mathbb{R}^3 : -N - 2N^{-1} \leq \xi_1 \leq -N + 2N^{-1}, -2 \leq \xi_2, \xi_3 \leq 2\}.$$

In order to treat  $\widehat{u_N \overline{u_N}}(\xi)$  one has to consider 4 terms. We have

$$\left| \mathcal{F}_x \left( \int_0^t e^{i(t-t')A^{1/2}} A^{-1/2} (e^{it'\Delta} \check{\chi}_{D_1} \overline{e^{it'\Delta} \check{\chi}_{D_2}}) dt' \right) (\xi) \right| \quad (14) \\ \sim \langle \xi \rangle^{-1} \left| \int_0^t \int_{\eta \in D_1, \eta - \xi \in D_2} e^{it'(\langle \xi \rangle - |\eta|^2 + |\xi - \eta|^2)} \chi_{D_1}(\eta) \chi_{D_2}(\eta - \xi) d\eta dt' \right|.$$

The inner integral vanishes unless

$$\xi \in D' := \{2N+1 - 3N^{-1} \leq \xi_1 \leq 2N+1 + 3N^{-1}, -3 \leq \xi_2, \xi_3 \leq 3\}.$$

Now for the phase factor we have for such  $\xi$ :

$$\langle \xi \rangle - |\eta|^2 + |\xi - \eta|^2 = 2N+1 - (N+1)^2 + N^2 + 0(1) = 0(1).$$

so that for  $|t| \ll 1$  we get  $|t'(\langle \xi \rangle - |\eta|^2 + |\xi - \eta|^2)| \ll 1$  and  $\langle \xi \rangle \sim 2N$ , and therefore

$$\int_0^t e^{it'(\langle \xi \rangle - |\eta|^2 + |\xi - \eta|^2)} dt' \sim t.$$

Moreover, if  $\eta \in D_1$  and  $\xi \in D$ , where

$$D := \{2N+1 - N^{-1} \leq \xi_1 \leq 2N+1 + N^{-1}, -1 \leq \xi_2, \xi_3 \leq 1\}$$

then automatically  $\eta - \xi \in D_2$ , so that for such  $\xi$  the region of integration over  $\eta$  is of size  $|D_1| \sim N^{-1}$ . Thus for  $\xi \in D$  we get:  $(14) \gtrsim |t|N^{-2}$ , so that integration over  $\xi \in D$  with  $|D| \sim N^{-1}$  gives

$$\left\| \int_0^t e^{i(t-t')A^{1/2}} A^{-1/2} (e^{it'\Delta} \tilde{\chi}_{D_1} \overline{e^{it'\Delta} \tilde{\chi}_{D_2}}) dt' \right\|_{H^\sigma(\mathbb{R}^3)} \gtrsim |t|N^{\sigma-\frac{5}{2}}. \quad (15)$$

Next we treat the term where the roles of  $D_1$  and  $D_2$  are exchanged. It vanishes unless

$$\xi \in \{-2N-1-3N^{-1} \leq \xi_1 \leq -2N-1+3N^{-1}, -3 \leq \xi_2, \xi_3 \leq 3\},$$

so that its support is disjoint to the support  $D'$  in the previous case. Similarly the term coming from the product of  $e^{it'\Delta} \tilde{\chi}_{D_1} \overline{e^{it'\Delta} \tilde{\chi}_{D_1}}$  and  $e^{it'\Delta} \tilde{\chi}_{D_2} \overline{e^{it'\Delta} \tilde{\chi}_{D_2}}$  vanishes unless  $\xi \in \{-2N^{-1} \leq \xi_1 \leq 2N^{-1}, -2 \leq \xi_2, \xi_3 \leq 2\}$  and  $\xi \in \{-4N^{-1} \leq \xi_1 \leq 4N^{-1}, -4 \leq \xi_2, \xi_3 \leq 4\}$ , respectively, so that these supports are also disjoint to  $D'$ . This implies

$$\left\| \int_0^t e^{-i(t-t')A^{1/2}} A^{-1/2} (e^{it'\Delta} u_N \overline{e^{it'\Delta} u_N}) dt' \right\|_{H^\sigma(\mathbb{R}^3)} \gtrsim |t|N^{\sigma-\frac{5}{2}}.$$

Combined with  $\|u_N\|_{H^s} \sim N^{s-\frac{1}{2}}$  this gives the claimed result  $\square$

**Proposition 4.4.** *Let  $0 < T \leq 1$ . For any  $N \gg 1$  there exists  $u_N \in H^s(\mathbb{R}^3)$  such that*

$$\sup_{|t| \leq T} \left\| \int_0^t e^{-i(t-t')A^{1/2}} A^{-1/2} (e^{it'\Delta} u_N \overline{e^{it'\Delta} u_N}) dt' \right\|_{H^1(\mathbb{R}^3)} \geq c_0 N^{-2s-\frac{1}{2}} \|u_N\|_{H^s(\mathbb{R}^3)}^2$$

for any  $\sigma \in \mathbb{R}$ , where  $c_0 > 0$  is independent of  $N$ .

*Proof.* Let  $\hat{u}_N := \chi_{D_1} + \chi_{D_2}$ , where

$$\begin{aligned} D_1 &:= \{\xi \in \mathbb{R}^3 : N - N^{-1} \leq \xi_1 \leq N + N^{-1}, -1 \leq \xi_2, \xi_3 \leq 1\}, \\ D_2 &:= \{\xi \in \mathbb{R}^3 : N + 4N^{-1} \leq \xi_1 \leq N + 7N^{-1}, -2 \leq \xi_2, \xi_3 \leq 2\}. \end{aligned}$$

We first consider the term (14). This vanishes unless  $\xi \in D'$ , where

$$D' := \{-8N^{-1} \leq \xi_1 \leq -3N^{-1}, -3 \leq \xi_2, \xi_3 \leq 3\},$$

so that for  $\xi \in D'$  we have  $|\langle \xi \rangle - |\eta|^2 + |\xi - \eta|^2| \lesssim 1$  and  $\langle \xi \rangle \sim 1$ , thus for  $|t| \ll 1$ :

$$\int_0^t e^{it'(\langle \xi \rangle - |\eta|^2 + |\xi - \eta|^2)} dt' \sim t.$$

Moreover, if  $\eta \in D_1$  and  $\xi \in D$ , then automatically  $\eta - \xi \in D_2$ , where

$$D := \{-6N^{-1} \leq \xi_1 \leq -5N^{-1}, -1 \leq \xi_2, \xi_3 \leq 1\},$$

so that for such  $\xi$  the region for the integration over  $\eta$  is of size  $|D_1| \sim N^{-1}$ , which implies the lower bound:

$$\left| \mathcal{F}_x \left( \int_0^t e^{i(t-t')A^{1/2}} A^{-1/2} (e^{it'\Delta} \tilde{\chi}_{D_1} \overline{e^{it'\Delta} \tilde{\chi}_{D_2}}) dt' \right) (\xi) \right| \gtrsim |t|N^{-1}, \quad (16)$$

so that integration over  $\xi \in D$  with  $|D| \sim N^{-1}$  and  $\langle \xi \rangle \sim 1$  gives:

$$\left\| \int_0^t e^{i(t-t')A^{1/2}} A^{-1/2} (e^{it'\Delta} \tilde{\chi}_{D_1} \overline{e^{it'\Delta} \tilde{\chi}_{D_2}}) dt' \right\|_{H^1(\mathbb{R}^3)} \gtrsim |t|N^{-\frac{3}{2}}.$$

Next we consider the integrals

$$\int e^{it'(\langle \xi \rangle - |\eta|^2 + |\xi - \eta|^2)} \chi_{D_j}(\eta) \chi_{D_m}(\eta - \xi) d\eta$$

for the other combinations of  $j, m \in \{1, 2\}$ . This term vanishes unless

- in the case  $j = 2, m = 1$ :  $\xi \in \{3N^{-1} \leq \xi_1 \leq 8N^{-1}, -3 \leq \xi_2, \xi_3 \leq 3\}$ ,

- in the case  $j = m = 1$ :  $\xi \in \{-2N^{-1} \leq \xi_1 \leq 2N^{-1}, -2 \leq \xi_2, \xi_3 \leq 2\}$ ,
- in the case  $j = m = 2$ :  $\xi \in \{-3N^{-1} \leq \xi_1 \leq 3N^{-1}, -4 \leq \xi_2, \xi_3 \leq 4\}$ .

In all the cases these sets are disjoint to  $D'$ , so that we conclude

$$\left\| \int_0^t e^{-i(t-t')A^{1/2}} A^{-1/2} (e^{it'\Delta} u_N \overline{e^{it'\Delta} u_N}) dt' \right\|_{H^\sigma(\mathbb{R}^3)} \geq c_0 t N^{-\frac{3}{2}}.$$

Combined with  $\|u_N\|_{H^s} \sim N^{s-\frac{1}{2}}$  this gives the claimed result.  $\square$

**Proposition 4.5.** *Let  $0 < T \leq 1$ . For all  $N \gg T^{-1}$  there exists  $u_N \in H^s(\mathbb{R}^3)$  and  $v_N \in H^\sigma(\mathbb{R}^3)$  such that*

$$\begin{aligned} \sup_{|t| \leq T} \left\| \int_0^t e^{-i(t-t')\Delta} (e^{it'\Delta} u_N \operatorname{Re}(e^{-it'A^{1/2}} v_N)) dt' \right\|_{H^s(\mathbb{R}^3)} \\ \geq c_0 N^{-\sigma-\frac{1}{2}} \|u_N\|_{H^s(\mathbb{R}^3)} \|v_N\|_{H^\sigma(\mathbb{R}^3)} \end{aligned}$$

for any  $s \in \mathbb{R}$ , where  $c_0 > 0$  is independent of  $N$ .

*Proof.* The proof of [3, Prop. 6.5] in two dimensions is also true in the three dimensional case with obvious modifications.  $\square$

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